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Research Article

Existence of Mild Solutions to Fractional Integrodifferential Equations of Neutral Type with Infinite Delay

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We study the solvability of the fractional integrodifferential equations of neutral type with infinite delay in a Banach space X. An existence result of mild solutions to such problems is obtained under the conditions in respect of Kuratowski's measure of noncompactness. As an application of the abstract result, we show the existence of solutions for an integrodifferential equation.

1. Introduction

The fractional differential equations are valuable tools in the modeling of many phenomena in various fields of science and engineering; so, they attracted many researchers (cf., e.g., [1–6] and references therein). On the other hand, the integrodifferential equations arise in various applications such as viscoelasticity, heat equations, and many other physical phenomena (cf., e.g., [7–10] and references therein). Moreover, the Cauchy problem for various delay equations in Banach spaces has been receiving more and more attention during the past decades (cf., e.g., [7, 10–15] and references therein).

Neutral functional differential equations arise in many areas of applied mathematics and for this reason, the study of this type of equations has received great attention in the last few years (cf., e.g., [12, 14–16] and references therein). In [12, 16], Hernández and Henríquez studied neutral functional differential equations with infinite delay. In the following, we will extend such results to fractional-order functional differential equations of neutral type with infinite delay. To the authors' knowledge, few papers can be found in the literature for the solvability of the fractional-order functional integrodifferential equations of neutral type with infinite delay.

In the present paper, we will consider the following fractional integrodifferential equation of neutral type with infinite delay in Banach space X:

$$\frac{d^q}{dt^q}(x(t)-h(t,x_t)) = A(x(t)-h(t,x_t)) + \int_0^t \beta(t,s)f(s,x(s),x_s)ds, \quad t \in [0,\ T],$$
$$x(t) = \phi(t) \in \mathcal{P}, \quad t \in (-\infty, 0], \tag{1.1}$$

where $T > 0$, $0 < q < 1$, $\mathcal{P}$ is a phase space that will be defined later (see Definition 2.5). A is a generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ of uniformly bounded linear operators on X. Then, there exists $M \geq 1$ such that $\|S(t)\| \leq M$. $h : [0,\ T] \times \mathcal{P} \to X$, $f : [0,\ T] \times X \times \mathcal{P} \to X$, $\beta : D \to \mathbf{R}$ ($D = \{(t,s) \in [0,\ T] \times [0,\ T] : t \geq s\}$), and $x_t : (-\infty,\ 0] \to X$ defined by $x_t(\theta) = x(t+\theta)$, for $\theta \in (-\infty,\ 0]$, ϕ belongs to $\mathcal{P}$ and $\phi(0) = 0$. The fractional derivative is understood here in the Caputo sense.

The aim of our paper is to study the solvability of (1.1) and present the existence of mild solution of (1.1) based on Kuratowski's measures of noncompactness. Moreover, an example is presented to show an application of the abstract results.

2. Preliminaries

Throughout this paper, we set $J := [0,\ T]$ and denote by X a real Banach space, by $L(X)$ the Banach space of all linear and bounded operators on X, and by $C(J,\ X)$ the Banach space of all X-valued continuous functions on J with the uniform norm topology.

Let us recall the definition of Kuratowski's measure of noncompactness.

Definition 2.1. Let B be a bounded subset of a seminormed linear space Y. Kuratowski's measure of noncompactness of B is defined as

$$\alpha(B) = \inf\{d > 0 : B \text{ has a finite cover by sets of diameter} \leq d\}. \tag{2.1}$$

This measure of noncompactness satisfies some important properties.

Lemma 2.2 (see [17]). *Let A and B be bounded subsets of X. Then,*

(1) $\alpha(A) \leq \alpha(B)$ *if* $A \subseteq B$,

(2) $\alpha(A) = \alpha(\overline{A})$, *where* $\overline{A}$ *denotes the closure of* A,

(3) $\alpha(A) = 0$ *if and only if* A *is precompact,*

(4) $\alpha(\lambda A) = |\lambda|\alpha(A)$, $\lambda \in \mathbf{R}$,

(5) $\alpha(A \cup B) = \max\{\alpha(A),\ \alpha(B)\}$,

(6) $\alpha(A+B) \leq \alpha(A) + \alpha(B)$, *where* $A + B = \{x + y : x \in A,\ y \in B\}$,

(7) $\alpha(A + a) = \alpha(A)$ *for any* $a \in X$,

(8) $\alpha(\overline{conv}A) = \alpha(A)$, *where* $\overline{conv}A$ *is the closed convex hull of* A.

For $H \subset C(J,\ X)$, we define

$$\int_0^t H(s)ds = \left\{ \int_0^t u(s)ds : u \in H \right\} \quad \text{for } t \in J, \tag{2.2}$$

where $H(s) = \{u(s) \in X : u \in H\}$.

The following lemmas will be needed.

Lemma 2.3 (see [17]). *If $H \subset C(J, X)$ is a bounded, equicontinuous set, then*

$$\alpha(H) = \sup_{t \in J} \alpha(H(t)). \tag{2.3}$$

Lemma 2.4 (see [18]). *If $\{u_n\}_{n=1}^{\infty} \subset L^1(J, X)$ and there exists an $m \in L^1(J, \mathbf{R}^+)$ such that $\|u_n(t)\| \leq m(t)$, a.e. $t \in J$, then $\alpha(\{u_n(t)\}_{n=1}^{\infty})$ is integrable and*

$$\alpha\left(\left\{ \int_0^t u_n(s)ds \right\}_{n=1}^{\infty} \right) \leq 2 \int_0^t \alpha(\{u_n(s)\}_{n=1}^{\infty}) ds. \tag{2.4}$$

The following definition about the phase space is due to Hale and Kato [11].

Definition 2.5. A linear space $\mathcal{P}$ consisting of functions from $\mathbf{R}^-$ into X with semi-norm $\|\cdot\|_{\mathcal{P}}$ is called an admissible phase space if $\mathcal{P}$ has the following properties.

(1) If $x : (-\infty,\ T] \to X$ is continuous on J and $x_0 \in \mathcal{P}$, then $x_t \in \mathcal{P}$ and x_t is continuous in $t \in J$ and

$$\|x(t)\| \leq C\|x_t\|_{\mathcal{P}}, \tag{2.5}$$

where $C \geq 0$ is a constant.

(2) There exist a continuous function $C_1(t) > 0$ and a locally bounded function $C_2(t) \geq 0$ in $t \geq 0$ such that

$$\|x_t\|_{\mathcal{P}} \leq C_1(t) \sup_{s \in [0,t]} \|x(s)\| + C_2(t)\|x_0\|_{\mathcal{P}}, \tag{2.6}$$

for $t \in [0,\ T]$ and x as in (1).

(3) The space $\mathcal{P}$ is complete.

Remark 2.6. (2.5) in (1) is equivalent to $\|\phi(0)\| \leq C\|\phi\|_{\mathcal{P}}$, for all $\phi \in \mathcal{P}$.

The following result will be used later.

Lemma 2.7 (see [19, 20]). *Let U be a bounded, closed, and convex subset of a Banach space X such that $0 \in U$, and let N be a continuous mapping of U into itself. If the implication*

$$V = \overline{conv} N(V) \ \text{or} \ V = N(V) \cup \{0\} \Longrightarrow \alpha(V) = 0 \tag{2.7}$$

holds for every subset V of U, then N has a fixed point.

Let Ω be a set defined by

$$\Omega = \left\{x : (-\infty,\ T] \longrightarrow X \text{ such that } x|_{(-\infty,\,0]} \in \mathcal{P},\ x|_J \in C(J,\ X)\right\}. \tag{2.8}$$

Motivated by [4, 5, 21], we give the following definition of mild solution of (1.1).

Definition 2.8. A function $x \in \Omega$ satisfying the equation

$$x(t) = \begin{cases} \phi(t), & t \in (-\infty,\ 0], \\ -Q(t)h(0,\phi) + h(t,x_t) + \displaystyle\int_0^t\int_0^s R(t-s)\beta(s,\tau)f(\tau,x(\tau),x_\tau)d\tau\, ds, & t \in J \end{cases} \tag{2.9}$$

is called a mild solution of (1.1), where

$$\begin{aligned} Q(t) &= \int_0^\infty \xi_q(\sigma)S(t^q\sigma)d\sigma, \\ R(t) &= q\int_0^\infty \sigma t^{q-1}\xi_q(\sigma)S(t^q\sigma)d\sigma \end{aligned} \tag{2.10}$$

and ξ_q is a probability density function defined on $(0,\ \infty)$ such that

$$\xi_q(\sigma) = \frac{1}{q}\sigma^{-1-(1/q)}\varpi_q\left(\sigma^{-1/q}\right) \geq 0, \tag{2.11}$$

where

$$\varpi_q(\sigma) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\sigma^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \quad \sigma \in (0,\infty). \tag{2.12}$$

Remark 2.9. According to [22], direct calculation gives that

$$\|R(t)\| \leq C_{q,M}t^{q-1}, \quad t > 0, \tag{2.13}$$

where $C_{q,M} = qM/\Gamma(1+q)$.

We list the following basic assumptions of this paper.

(H1) $f : J \times X \times \mathcal{P} \to X$ satisfies $f(\cdot,v,w) : J \to X$ is measurable, for all $(v,w) \in X \times \mathcal{P}$ and $f(t,\cdot,\cdot) : X \times \mathcal{P} \to X$ is continuous for a.e. $t \in J$, and there exist two positive functions $\mu_i(\cdot) \in L^1(J,\mathbf{R}^+)$ $(i = 1,2)$ such that

$$\|f(t,v,w)\| \leq \mu_1(t)\|v\| + \mu_2(t)\|w\|_{\mathcal{P}}, \quad (t,v,w) \in J \times X \times \mathcal{P}. \tag{2.14}$$

(H2) For any bounded sets $D_1 \subset X$, $D_2 \subset \mathcal{P}$, and $0 \le s \le t \le T$, there exists an integrable positive function η such that

$$\alpha\big(R(t-s)f(\tau, D_1, D_2)\big) \le \eta_t(s,\tau)\left(\alpha(D_1) + \sup_{-\infty<\theta\le 0} \alpha(D_2(\theta))\right), \tag{2.15}$$

where $\eta_t(s,\tau) := \eta(t,s,\tau)$ and $\sup_{t\in J}\int_0^t\int_0^s \eta_t(s,\tau)d\tau ds := \eta^* < \infty$.

(H3) There exists a constant $L > 0$ such that

$$\|h(t_1,\varphi) - h(t_2,\widetilde{\varphi})\| \le L\Big(|t_1 - t_2| + \|\varphi - \widetilde{\varphi}\|_{\mathcal{P}}\Big), \quad t_1, t_2 \in J,\ \varphi, \widetilde{\varphi} \in \mathcal{P}. \tag{2.16}$$

(H4) For each $t \in J$, $\beta(t,s)$ is measurable on $[0,t]$ and $\beta(t) = \operatorname{ess\,sup}\{|\beta(t,s)|, 0 \le s \le t\}$ is bounded on J. The map $t \to B_t$ is continuous from J to $L^\infty(J, \mathbf{R})$, here, $B_t(s) = \beta(t,s)$.

(H5) There exists $M^* \in (0,1)$ such that

$$LC_1^* + \frac{T^q\beta C_{q,M}}{q}\Big(\|\mu_1\|_{L^1(J,\,\mathbf{R}^+)} + C_1^*\|\mu_2\|_{L^1(J,\,\mathbf{R}^+)}\Big) < M^*, \tag{2.17}$$

where $C_1^* = \sup_{t\in J} C_1(t)$, $\beta = \sup_{t\in J}\beta(t)$.

3. Main Result

In this section, we will apply Lemma 2.7 to show the existence of mild solution of (1.1). To this end, we consider the operator $\Phi : \Omega \to \Omega$ defined by

$$(\Phi x)(t) = \begin{cases} \phi(t), & t\in(-\infty,\ 0], \\ -Q(t)h(0,\phi) + h(t,x_t) + \displaystyle\int_0^t\int_0^s R(t-s)\beta(s,\tau)f(\tau,\ x(\tau), x_\tau)d\tau\, ds, & t \in J. \end{cases} \tag{3.1}$$

It follows from (H1), (H3), and (H4) that Φ is well defined.

It will be shown that Φ has a fixed point, and this fixed point is then a mild solution of (1.1).

Let $\overline{y}(\cdot) : (-\infty, T] \to X$ be the function defined by

$$\overline{y}(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ 0, & t \in J. \end{cases} \tag{3.2}$$

Set $x(t) = \overline{y}(t) + z(t)$, $t \in (-\infty, T]$.

It is clear to see that x satisfies (2.9) if and only if z satisfies $z_0 = 0$ and for $t \in J$,

$$z(t) = -Q(t)h(0,\phi) + h(t,\overline{y}_t + z_t) + \int_0^t \int_0^s R(t-s)\beta(s,\tau)f(\tau,\overline{y}(\tau) + z(\tau),\overline{y}_\tau + z_\tau)d\tau\, ds. \tag{3.3}$$

Let $Z_0 = \{z \in \Omega : z_0 = 0\}$. For any $z \in Z_0$,

$$\|z\|_{Z_0} = \sup_{t\in J}\|z(t)\| + \|z_0\|_{\mathcal{P}} = \sup_{t\in J}\|z(t)\|. \tag{3.4}$$

Thus, $(Z_0, \ \|\cdot\|_{Z_0})$ is a Banach space. Set

$$B_r = \{z \in Z_0 : \|z\|_{Z_0} \le r\}, \quad \text{for some } r > 0. \tag{3.5}$$

Then, for $z \in B_r$, from(2.6), we have

$$\begin{aligned}
\|\overline{y}_t + z_t\|_{\mathcal{P}} &\le \|\overline{y}_t\|_{\mathcal{P}} + \|z_t\|_{\mathcal{P}} \\
&\le C_1(t)\sup_{0\le\tau\le t}\|\overline{y}(\tau)\| + C_2(t)\|\overline{y}_0\|_{\mathcal{P}} + C_1(t)\sup_{0\le\tau\le t}\|z(\tau)\| + C_2(t)\|z_0\|_{\mathcal{P}} \\
&= C_2(t)\|\phi\|_{\mathcal{P}} + C_1(t)\sup_{0\le\tau\le t}\|z(\tau)\| \\
&\le C_2^* \cdot \|\phi\|_{\mathcal{P}} + C_1^* r := r^*,
\end{aligned} \tag{3.6}$$

where $C_2^* = \sup_{0\le\eta\le T} C_2(\eta)$.

In order to apply Lemma 2.7 to show that Φ has a fixed point, we let $\widetilde{\Phi} : Z_0 \to Z_0$ be an operator defined by $(\widetilde{\Phi}z)(t) = 0, \ t \in (-\infty, 0]$ and for $t \in J$,

$$\begin{aligned}
(\widetilde{\Phi}z)(t) = &-Q(t)h(0,\phi) + h(t,\overline{y}_t + z_t) \\
&+ \int_0^t \int_0^s R(t-s)\beta(s,\tau)f(\tau,\overline{y}(\tau) + z(\tau),\overline{y}_\tau + z_\tau)d\tau\, ds.
\end{aligned} \tag{3.7}$$

Clearly, the operator Φ has a fixed point is equivalent to $\widetilde{\Phi}$ has one. So, it turns out to prove that $\widetilde{\Phi}$ has a fixed point.

Now, we present and prove our main result.

Theorem 3.1. *Assume that (H1)–(H5) are satisfied, then there exists a mild solution of* (1.1) *on* $(-\infty, \ T]$ *provided that* $L + 16\beta\eta^* < 1$.

Proof. For $z \in B_r$, $t \in J$, from (3.6), we have

$$\begin{aligned}
\|f(t,\overline{y}(t) + z(t),\overline{y}_t + z_t)\| &\le \mu_1(t)\|\overline{y}(t) + z(t)\| + \mu_2(t)\|\overline{y}_t + z_t\|_{\mathcal{P}} \\
&\le \mu_1(t)r + \mu_2(t)r^*.
\end{aligned} \tag{3.8}$$

In view of (H3),

$$\begin{aligned}\|h(t,\overline{y}_t+z_t)\| &\le \|h(t,\ \overline{y}_t+z_t)-h(t,0)\|+\|h(t,0)\| \\ &\le L\|\overline{y}_t+z_t\|_{\mathcal{P}}+M_1 \\ &\le Lr^*+M_1,\end{aligned} \tag{3.9}$$

where $M_1=\sup_{t\in J}\|h(t,0)\|$.

Next, we show that there exists some $r>0$ such that $\widetilde{\Phi}(B_r)\subset B_r$. If this is not true, then for each positive number r, there exist a function $z^r(\cdot)\in B_r$ and some $t\in J$ such that $\|(\widetilde{\Phi}z^r)(t)\|>r$. However, on the other hand, we have from (3.8), (3.9), and (H4)

$$\begin{aligned}r &< \left\|\left(\widetilde{\Phi}z^r\right)(t)\right\| \\ &\le \|-Q(t)h(0,\phi)\|+\|h(t,\overline{y}_t+z_t^r)\| \\ &\quad+\int_0^t\int_0^s\|R(t-s)\beta(s,\tau)f(\tau,\overline{y}(\tau)+z^r(\tau),\overline{y}_\tau+z_\tau^r)\|d\tau\,ds \\ &\le LM\|\phi\|_{\mathcal{P}}+MM_1+Lr^*+M_1+\beta C_{q,M}\int_0^t\int_0^s(t-s)^{q-1}\left[\mu_1(\tau)r+\mu_2(\tau)r^*\right]d\tau\,ds \\ &\le LM\|\phi\|_{\mathcal{P}}+MM_1+Lr^*+M_1+\beta rC_{q,M}\int_0^t\int_0^s(t-s)^{q-1}\mu_1(\tau)d\tau\,ds \\ &\quad+\beta r^*C_{q,M}\int_0^t\int_0^s(t-s)^{q-1}\mu_2(\tau)d\tau\,ds \\ &\le L\left(M\|\phi\|_{\mathcal{P}}+r^*\right)+M_1(M+1)+\frac{T^q\beta C_{q,M}}{q}\left[r\|\mu_1\|_{L^1(J,\mathbf{R}^+)}+r^*\|\mu_2\|_{L^1(J,\mathbf{R}^+)}\right].\end{aligned} \tag{3.10}$$

Dividing both sides of (3.10) by r, and taking $r\to\infty$, we have

$$LC_1^*+\frac{T^q\beta C_{q,M}}{q}\left(\|\mu_1\|_{L^1(J,\mathbf{R}^+)}+C_1^*\|\mu_2\|_{L^1(J,\mathbf{R}^+)}\right)\ge 1. \tag{3.11}$$

This contradicts (2.17). Hence, for some positive number r, $\widetilde{\Phi}(B_r)\subset B_r$.

Let $\{z^k\}_{k\in\mathbf{N}}\subset B_r$ with $z^k\to z$ in B_r as $k\to\infty$. Since f satisfies (H1), for almost every $t\in J$, we get

$$f\left(t,\overline{y}(t)+z^k(t),\overline{y}_t+z_t^k\right)\longrightarrow f(t,\overline{y}(t)+z(t),\overline{y}_t+z_t),\quad \text{as}\quad k\to\infty. \tag{3.12}$$

In view of (3.6), we have

$$\left\|\overline{y}_t + z_t^k\right\|_p \leq r^*. \tag{3.13}$$

Noting that

$$\left\|f\left(t, \overline{y}(t) + z^k(t), \overline{y}_t + z_t^k\right) - f\left(t, \overline{y}(t) + z(t), \overline{y}_t + z_t\right)\right\| \leq 2\mu_1(t)r + 2\mu_2(t)r^*, \tag{3.14}$$

we have by the Lebesgue Dominated Convergence Theorem that

$$\begin{aligned}
&\left\|\left(\widetilde{\Phi}z^k\right)(t) - \left(\widetilde{\Phi}z\right)(t)\right\| \\
&\quad \leq \left\|h\left(t, \overline{y}_t + z_t^k\right) - h\left(t, \overline{y}_t + z_t\right)\right\| \\
&\qquad + \int_0^t \int_0^s \left\|R(t-s)\beta(s,\tau)\left[f\left(\tau, \overline{y}(\tau) + z^k(\tau), \overline{y}_\tau + z_\tau^k\right) - f\left(\tau, \overline{y}(\tau) + z(\tau), \overline{y}_\tau + z_\tau\right)\right]\right\| d\tau\, ds \\
&\quad \leq L\left\|z_t^k - z_t\right\|_p \\
&\qquad + \beta C_{q,M} \int_0^t \int_0^s (t-s)^{q-1}\left\|f\left(\tau, \overline{y}(\tau) + z^k(\tau), \overline{y}_\tau + z_\tau^k\right) - f\left(\tau, \overline{y}(\tau) + z(\tau), \overline{y}_\tau + z_\tau\right)\right\| d\tau\, ds \\
&\quad \longrightarrow 0, \quad k \longrightarrow \infty.
\end{aligned} \tag{3.15}$$

Therefore, we obtain

$$\lim_{k\to\infty}\left\|\widetilde{\Phi}z^k - \widetilde{\Phi}z\right\|_{Z_0} = 0. \tag{3.16}$$

This shows that $\widetilde{\Phi}$ is continuous.

Set

$$G\left(\cdot, \overline{y}(\cdot) + z(\cdot), \overline{y}_{(\cdot)} + z_{(\cdot)}\right) := \int_0^{\cdot} \beta(\cdot,\tau) f\left(\tau, \overline{y}(\tau) + z(\tau), \overline{y}_\tau + z_\tau\right) d\tau. \tag{3.17}$$

Let $0 < t_2 < t_1 < T$ and $z \in B_r$, then we can see

$$\left\|\left(\widetilde{\Phi}z\right)(t_1) - \left(\widetilde{\Phi}z\right)(t_2)\right\| \leq I_1 + I_2 + I_3 + I_4, \tag{3.18}$$

where

$$
\begin{aligned}
I_1 &= \|Q(t_1) - Q(t_2)\| \cdot \|h(0,\phi)\|,\\
I_2 &= \left\|h\left(t_1, \overline{y}_{t_1} + z_{t_1}\right) - h\left(t_2, \overline{y}_{t_2} + z_{t_2}\right)\right\|,\\
I_3 &= \left\|\int_0^{t_2} [R(t_1 - s) - R(t_2 - s)] G\left(s, \overline{y}(s) + z(s),\ \overline{y}_s + z_s\right) ds\right\|,\\
I_4 &= \int_{t_2}^{t_1} \|R(t_1 - s)\| \left\|G\left(s, \overline{y}(s) + z(s), \overline{y}_s + z_s\right)\right\| ds.
\end{aligned}
\tag{3.19}
$$

It follows the continuity of $S(t)$ in the uniform operator topology for $t > 0$ that I_1 tends to 0, as $t_2 \to t_1$. The continuity of h ensures that I_2 tends to 0, as $t_2 \to t_1$.

For I_3, we have

$$
\begin{aligned}
I_3 &\le q \int_0^{t_2} \int_0^{\infty} \sigma \left\| \left[(t_1 - s)^{q-1} - (t_2 - s)^{q-1}\right] \xi_q(\sigma) S\left((t_1 - s)^q \sigma\right) G\left(s, \overline{y}(s) + z(s), \overline{y}_s + z_s\right) \right\| d\sigma ds\\
&\quad + q \int_0^{t_2} \int_0^{\infty} \sigma (t_2 - s)^{q-1} \xi_q(\sigma) \left\| S\left((t_1 - s)^q \sigma\right) - S\left((t_2 - s)^q \sigma\right) \right\|\\
&\qquad \times \left\| G\left(s, \overline{y}(s) + z(s), \overline{y}_s + z_s\right) \right\| d\sigma\, ds\\
&\le C_{q,M} \int_0^{t_2} \left| (t_1 - s)^{q-1} - (t_2 - s)^{q-1} \right| \left\| G\left(s, \overline{y}(s) + z(s), \overline{y}_s + z_s\right) \right\| ds\\
&\quad + q \int_0^{t_2} \int_0^{\infty} \sigma (t_2 - s)^{q-1} \xi_q(\sigma) \left\| S\left((t_1 - s)^q \sigma\right) - S\left((t_2 - s)^q \sigma\right) \right\|\\
&\qquad \times \left\| G\left(s, \overline{y}(s) + z(s), \overline{y}_s + z_s\right) \right\| d\sigma\, ds,\\
&\le \beta\left[r \|\mu_1\|_{L^1(J,\mathbf{R}^+)} + r^* \|\mu_2\|_{L^1(J,\mathbf{R}^+)} \right]\\
&\quad \times \left[C_{q,M} \int_0^{t_2} \left| (t_1 - s)^{q-1} - (t_2 - s)^{q-1} \right| ds \right.\\
&\qquad \left. + q \int_0^{t_2} \int_0^{\infty} \sigma (t_2 - s)^{q-1} \xi_q(\sigma) \left\| S\left((t_1 - s)^q \sigma\right) - S\left((t_2 - s)^q \sigma\right) \right\| d\sigma\, ds \right].
\end{aligned}
\tag{3.20}
$$

Clearly, the first term on the right-hand side of (3.20) tends to 0 as $t_2 \to t_1$. The second term on the right-hand side of (3.20) tends to 0 as $t_2 \to t_1$ as a consequence of the continuity of $S(t)$ in the uniform operator topology for $t > 0$.

In view of the assumption of $\mu_i(s)$ $(i = 1, 2)$ and (3.8), we see that

$$\begin{aligned} I_4 &\le C_{q,M} \int_{t_2}^{t_1} (t_1 - s)^{q-1} \|G(s, \overline{y}(s) + z(s), \overline{y}_s + z_s)\| ds \\ &\le \beta C_{q,M} \left[r\|\mu_1\|_{L^1(J,\mathbf{R}^+)} + r^* \|\mu_2\|_{L^1(J,\mathbf{R}^+)} \right] \int_{t_2}^{t_1} (t_1 - s)^{q-1} ds \\ &\longrightarrow 0, \quad \text{as } t_2 \longrightarrow t_1. \end{aligned} \tag{3.21}$$

Thus, $\widetilde{\Phi}(B_r)$ is equicontinuous.

Now, let V be an arbitrary subset of B_r such that $V \subset \overline{conv}(\widetilde{\Phi}(V) \cup \{0\})$.

Set $(\widetilde{\Phi}_1 z)(t) = h(t, \overline{y}_t + z_t)$,

$$\left(\widetilde{\Phi}_2 z\right)(t) = -Q(t) h(0, \phi) + \int_0^t \int_0^s R(t-s) \beta(s, \tau) f(\tau, \overline{y}(\tau) + z(\tau), \overline{y}_\tau + z_\tau) d\tau\, ds. \tag{3.22}$$

Noting that for $z, \widetilde{z} \in V$, we have

$$\|h(t, \overline{y}_t + \widetilde{z}_t) - h(t, \overline{y}_t + z_t)\| \le L\|\widetilde{z}_t - z_t\|_{\mathcal{P}}. \tag{3.23}$$

Thus,

$$\alpha(h(t, \overline{y}_t + V_t)) \le L\alpha(V_t) \le L \sup_{-\infty<\theta\le 0} \alpha(V(t+\theta)) = L \sup_{0\le\tau\le t} \alpha(V(\tau)) \le L\alpha(V), \tag{3.24}$$

where $V_t = \{z_t : z \in V\}$. Therefore, $\alpha(\widetilde{\Phi}_1 V) = \sup_{t\in J} \alpha((\widetilde{\Phi}_1 V)(t)) \le L\alpha(V)$.

Moreover, for any $\varepsilon > 0$ and bounded set D, we can take a sequence $\{v_n\}_{n=1}^{\infty} \subset D$ such that $\alpha(D) \le 2\alpha(\{v_n\}) + \varepsilon$ (see [23], P125). Thus, for $\{v_n\}_{n=1}^{\infty} \subset V$, noting that the choice of V, and from Lemmas 2.2–2.4 and (H2), we have

$$
\begin{aligned}
\alpha\left(\widetilde{\Phi}_2 V\right) &\le 2\alpha\left(\left\{\widetilde{\Phi}_2 v_n\right\}\right)+\varepsilon = 2\sup_{t\in J}\alpha\left(\left\{\widetilde{\Phi}_2 v_n(t)\right\}\right)+\varepsilon \\
&= 2\sup_{t\in J}\alpha\left(\left\{\int_0^t R(t-s)\int_0^s \beta(s,\tau)f\left(\tau,\overline{y}(\tau)+v_n(\tau),\overline{y}_\tau+v_{n\tau}\right)d\tau ds\right\}\right)+\varepsilon \\
&\le 4\sup_{t\in J}\int_0^t \alpha\left(\left\{R(t-s)\int_0^s \beta(s,\tau)f\left(\tau,\overline{y}(\tau)+v_n(\tau),\overline{y}_\tau+v_{n\tau}\right)d\tau\right\}\right)ds+\varepsilon \\
&\le 8\sup_{t\in J}\int_0^t\int_0^s \alpha\left(\left\{R(t-s)\beta(s,\tau)f\left(\tau,\overline{y}(\tau)+v_n(\tau),\overline{y}_\tau+v_{n\tau}\right)\right\}\right)d\tau\, ds+\varepsilon \\
&\le 8\beta\sup_{t\in J}\int_0^t\int_0^s \alpha\left(\left\{R(t-s)f\left(\tau,\overline{y}(\tau)+v_n(\tau),\overline{y}_\tau+v_{n\tau}\right)\right\}\right)d\tau\, ds+\varepsilon \\
&\le 8\beta\sup_{t\in J}\int_0^t\int_0^s \eta_t(s,\tau)\left[\alpha(\{v_n(\tau)\})+\sup_{-\infty<\theta\le 0}\alpha(\{v_n(\theta+\tau)\})\right]d\tau\, ds+\varepsilon \\
&\le 8\beta\sup_{t\in J}\int_0^t\int_0^s \eta_t(s,\tau)\left[\alpha(\{v_n\})+\sup_{0\le\mu\le\tau}\alpha\left(\{v_n(\mu)\}\right)\right]d\tau\, ds+\varepsilon \\
&\le 16\beta\alpha(\{v_n\})\sup_{t\in J}\int_0^t\int_0^s \eta_t(s,\tau)d\tau\, ds+\varepsilon \le 16\beta\eta^*\alpha(V)+\varepsilon.
\end{aligned}
\tag{3.25}
$$

It follows from Lemma 2.2 that

$$
\alpha(V)\le\alpha\left(\widetilde{\Phi}V\right)\le\alpha\left(\widetilde{\Phi}_1 V\right)+\alpha\left(\widetilde{\Phi}_2 V\right)\le\left(L+16\beta\eta^*\right)\alpha(V)+\varepsilon, \tag{3.26}
$$

since ε is arbitrary, we can obtain

$$
\alpha(V)\le\left(L+16\beta\eta^*\right)\alpha(V). \tag{3.27}
$$

Hence, $\alpha(V)=0$. Applying now Lemma 2.7, we conclude that $\widetilde{\Phi}$ has a fixed point z^* in B_r. Let $x(t)=\overline{y}(t)+z^*(t)$, $t\in(-\infty,T]$, then $x(t)$ is a fixed point of the operator Φ which is a mild solution of (1.1). □

4. Application

In this section, we consider the following integrodifferential model:

$$\begin{aligned}
&\frac{\partial^q}{\partial t^q}\left[v(t,\xi)-t\int_{-\infty}^{0}\gamma_1(\theta)\frac{|v(t+\theta,\xi)|}{1+|v(t+\theta,\xi)|}d\theta\right]\\
&\quad=\frac{\partial^2}{\partial \xi^2}\left[v(t,\xi)-t\int_{-\infty}^{0}\gamma_1(\theta)\frac{|v(t+\theta,\xi)|}{1+|v(t+\theta,\xi)|}d\theta\right]\\
&\qquad+\int_0^t (t-s)\frac{s^k}{k}\sin|v(s,\xi)|\cdot\int_0^s \cos v(\tau,\xi)\,d\tau\, ds\\
&\qquad+\int_0^t (t-s)\int_{-\infty}^{0}\gamma_2(\theta)\sin\left(s^2|v(s+\theta,\xi)|\right)d\theta\, ds,\\
&v(t,0)-t\int_{-\infty}^{0}\gamma_1(\theta)\frac{|v(t+\theta,0)|}{1+|v(t+\theta,0)|}d\theta=0,\\
&v(t,1)-t\int_{-\infty}^{0}\gamma_1(\theta)\frac{|v(t+\theta,1)|}{1+|v(t+\theta,1)|}d\theta=0,\\
&v(\theta,\xi)=v_0(\theta,\xi),\quad -\infty<\theta\leq 0,
\end{aligned} \tag{4.1}$$

where $0 \leq t \leq 1$, $\xi \in [0,1]$, $k \in \mathbf{N}$, $\gamma_1,\gamma_2 : (-\infty,0] \to \mathbf{R}$, $v_0 : (-\infty,0]\times[0,1] \to \mathbf{R}$ are continuous functions, and $\int_{-\infty}^{0}|\gamma_i(\theta)|d\theta < \infty (i = 1,2)$.

Set $X = L^2([0,1],\mathbf{R})$ and define A by

$$\begin{gathered}
D(A) = H^2(0,1)\cap H_0^1(0,1),\\
Au = u''.
\end{gathered} \tag{4.2}$$

Then, A generates a compact, analytic semigroup $S(\cdot)$ of uniformly bounded, linear operators, and $\|S(t)\| \leq 1$.

Let the phase space $\mathcal{D}$ be $BUC(\mathbf{R}^-,X)$, the space of bounded uniformly continuous functions endowed with the following norm:

$$\|\varphi\|_{\mathcal{D}} = \sup_{-\infty<\theta\leq 0}\left|\varphi(\theta)\right|,\quad \forall\,\varphi\in\mathcal{D}, \tag{4.3}$$

then we can see that $C_1(t) = 1$ in (2.6).

For $t \in [0,1]$, $\xi \in [0,\ 1]$ and $\varphi \in BUC(\mathbf{R}^-, X)$, we set

$$
\begin{gathered}
x(t)(\xi) = v(t,\xi), \\
\phi(\theta)(\xi) = v_0(\theta,\xi), \quad \theta \in (-\infty, 0], \\
h(t,\varphi)(\xi) = t\int_{-\infty}^{0} \gamma_1(\theta)\frac{|\varphi(\theta)(\xi)|}{1+|\varphi(\theta)(\xi)|}d\theta, \\
\beta(t,s) = t-s, \\
f(t,x(t),\varphi)(\xi) = \frac{t^k}{k}\sin|x(t)(\xi)|\cdot\int_0^t \cos x(s)(\xi)ds + \int_{-\infty}^{0}\gamma_2(\theta)\sin\left(t^2|\varphi(\theta)(\xi)|\right)d\theta.
\end{gathered}
\tag{4.4}
$$

Then (4.1) can be reformulated as the abstract (1.1).

Moreover, for $t \in [0,\ 1]$, we can see

$$
\begin{aligned}
\|f(t,x(t),\varphi)(\xi)\| &\le \frac{t^{k+1}}{k}\|x(t)\| + t^2\|\varphi\|_{\mathcal{P}}\int_{-\infty}^{0}|\gamma_2(\theta)|d\theta \\
&= \mu_1(t)\|x(t)\| + \mu_2(t)\|\varphi\|_{\mathcal{P}},
\end{aligned}
\tag{4.5}
$$

where $\mu_1(t) := t^{k+1}/k$, $\mu_2(t) := t^2\int_{-\infty}^{0}|\gamma_2(\theta)|d\theta$.

For $t_1, t_2 \in [0,1]$, $\varphi,\ \widetilde{\varphi} \in \mathcal{P}$, we have

$$
\begin{aligned}
\|h(t_1,\varphi) - h(t_2,\widetilde{\varphi})\| &\le |t_1 - t_2|\int_{-\infty}^{0}\left\|\gamma_1(\theta)\frac{|\varphi(\theta)(\xi)|}{1+|\varphi(\theta)(\xi)|}\right\|d\theta \\
&\quad + t_2\int_{-\infty}^{0}\left\|\gamma_1(\theta)\left(\frac{|\varphi(\theta)(\xi)|}{1+|\varphi(\theta)(\xi)|} - \frac{|\widetilde{\varphi}(\theta)(\xi)|}{1+|\widetilde{\varphi}(\theta)(\xi)|}\right)\right\|d\theta \\
&\le |t_1 - t_2|\int_{-\infty}^{0}|\gamma_1(\theta)|d\theta + \int_{-\infty}^{0}|\gamma_1(\theta)|d\theta\cdot\|\varphi - \widetilde{\varphi}\|_{\mathcal{P}} \\
&= L\left(|t_1 - t_2| + \|\varphi - \widetilde{\varphi}\|_{\mathcal{P}}\right),
\end{aligned}
\tag{4.6}
$$

where $L = \int_{-\infty}^{0}|\gamma_1(\theta)|d\theta$.

Suppose further that there exists a constant $M^* \in (0,1)$ such that

$$
L + \frac{C_{q,M}}{q}\left(\|\mu_1\|_{L^1([0,1],\mathbf{R}^+)} + \|\mu_2\|_{L^1([0,1],\mathbf{R}^+)}\right) < M^*,
\tag{4.7}
$$

then (4.1) has a mild solution by Theorem 3.1.

For example, if we put

$$
\gamma_1(\theta) = \gamma_2(\theta) = e^{k\theta}, \quad q = 0.5, \quad k = 2,
\tag{4.8}
$$

then $L = 1/2$, $C_{q,M} = 1/\Gamma(0.5) = 1/\sqrt{\pi}$, $\|\mu_1\|_{L^1([0,1],\mathbf{R}^+)} = 1/8$, $\|\mu_2\|_{L^1([0,1],\mathbf{R}^+)} = 1/6$. Thus, we see

$$L + \frac{C_{q,M}}{q}\left(\|\mu_1\|_{L^1([0,1],\mathbf{R}^+)} + \|\mu_2\|_{L^1([0,1],\mathbf{R}^+)}\right) = \frac{1}{2} + \frac{2}{\sqrt{\pi}}\left(\frac{1}{8} + \frac{1}{6}\right) < 0.9 < 1. \tag{4.9}$$

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